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EXACT SOLUTIONS OF PROBLEMS ON HARMONIC VIBRATIONS OF A THERMOELASTIC ROD HAVING A TRIANGULAR CROSS SECTION WITH ACCOUNT FOR THE CONNECTEDNESS

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Two exact solutions of the problem on harmonic vibrations of a thermoelastic rod with a cross section representing a right triangle have been obtained with the use of multiaction logic operations. The influence of the connectedness of the problem as well as the temperature and elastic properties of the indicated rod on the wave process of its deformation has been investigated. Expressions for the velocities of the temperature, longitudinal, and shear waves were obtained. A criterion M_0 for the expediency of taking into account the connectedness in the formulation of the problem was determined.

Introduction. Of the known solutions of multidimensional problems on vibrations of bodies, we considered the solution obtain in [1], where a connected model of heat propagation with a finite rate was considered, the solution obtained in [2] with the use of the boundary-element method, and the solution of the problem on the heating of a cylinder by the friction of a band against its surface, obtained in [3].

Formulation of the Problem. In many cases, temperature fields interact with elastic deformations, and this interaction can be significant. Metals and their alloys exhibit thermoelastic properties under small mechanical and heat loads. Solid bodies with such complex properties can be defined by different rheological models. For definiteness, we will use a model in which the total deformation consists of elastic and thermal deformations. The stress tensor is expressed in terms of the deformation tensor and the temperature with the use of the Duhamel–Neumann dependences [4]:

$$\sigma_{ij} = \lambda\,(e_{kk} - 3\alpha_t T)\,\delta_{ij} + 2\mu\,(e_{ij} - \alpha_t T\delta_{ij})\,. \tag{1}$$

From this point on we will solve the dynamic problem under the plane-deformation conditions. Substituting σ_{ij} from (1) into the equation of continuous-medium motion and using the heat-conduction equation, we obtain three differential equations for the displacements u and v in the Cartesian coordinate system and the temperature T:

$$\begin{gathered}\lambda_0 u_{xx} + (\lambda + \mu)\,v_{xy} + \mu u_{yy} - \gamma T_x = \rho u_{tt}\,, \quad \lambda_0 = \lambda + 2\mu\,,\\ \lambda_0 v_{yy} + (\lambda + \mu)\,u_{xy} + \mu v_{xx} - \gamma T_y = \rho v_{tt}\,, \quad \gamma = (3\lambda + 2\mu)\,\alpha_t\,,\\ b\Delta T - k\,(u_{xt} + v_{yt}) = T_t\,, \quad k = \gamma T_0/C\rho\,,\\ (x, y) \in \Omega\,, \quad 0 \le t \le t_0\,, \quad (u, v, t) \in C^{(2)}\,(\Omega \times [0, t_0])\,.\end{gathered} \tag{2}$$

The region Ω with a boundary Γ is a right triangle of height h. The quantity k in (2) is called the connectivity coefficient, and the corresponding term in the heat-conduction equation accounts for the effect of change in the temperature of a solid body as a result of an adiabatic change in its volume. In the case where $k = 0$, the heat-conduction equation becomes independent of the equation of motion and can be solved independently, e.g., by the known method proposed in [5]. At $k \neq 0$, the problem becomes much more complex because system (2) becomes connected and each

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of the three equations involves the unknown functions u, v, and T. Naturally, this brings up the following questions: In which cases should the connectedness be taken into account? What are the properties on which the connectedness has a significant influence? and Is a resonance possible? To answer these questions, we will analyze exact solutions of geometrically two-dimensional problems. The following boundary conditions are set for Eqs. (2) in the region Ω with a boundary Γ:

$$u_n\big|_\Gamma = u_{10}\cos\omega t + u_{20}\sin\omega t\,,\quad \tau_n\big|_\Gamma = \tau_{10}\cos\omega t + \tau_{20}\sin\omega t\,,\quad \partial T/\partial n\big|_\Gamma = q_{10}\cos\omega t + q_{20}\sin\omega t\,; \tag{3}$$

$$\sigma_n\big|_\Gamma = \sigma_{10}\cos\omega t + \sigma_{20}\sin\omega t\,,\quad u_\tau\big|_\Gamma = v_{10}\cos\omega t + v_{20}\sin\omega t\,,\quad T\big|_\Gamma = T_{10}\cos\omega t + T_{20}\sin\omega t\,. \tag{4}$$

We will consider the problem on harmonic vibrations with no initial condition and represent the unknown quantities u, v, and T in the form

$$u = U_1(x, y)\cos\omega t + U_2(x, y)\sin\omega t\,,\quad v = V_1(x, y)\cos\omega t + V_2(x, y)\sin\omega t\,,$$
$$T = T_1(x, y)\cos\omega t + T_2(x, y)\sin\omega t\,. \tag{5}$$

Substitution of (5) into (2) gives

$$\lambda_0 U_{jxx} + (\lambda + \mu)\, V_{jxy} + \mu U_{jyy} - \gamma T_{jx} + \rho\omega^2 U_j = 0\,,\quad j = 1, 2\,;$$
$$\lambda_0 U_{jyy} + (\lambda + \mu)\, U_{jxy} + \mu V_{jxx} - \gamma T_{jy} + \rho\omega^2 V_j = 0\,, \tag{6}$$
$$b\Delta T_1 - \omega k\,(U_{2x} + V_{2y}) - \omega T_2 = 0\,,\quad b\Delta T_2 + \omega k\,(U_{1x} + V_{1y}) + \omega T_1 = 0\,.$$

One-Dimensional Solution. At first, for Eqs. (6), we will consider an analogous simpler auxiliary problem with no boundary conditions for a plane layer in the case where U_j, V_j, and T_j (j = 1, 2) are determined only by the coordinate x. In line with this assumption, the following designations are introduced:

$$U_j = P_j(x)\,,\quad V_j = Q_j(x)\,,\quad T_j = R_j(x)\,,\quad j = 1, 2\,. \tag{7}$$

In this case, system (6) takes the form of ordinary differential equations

$$\lambda_0 P_1'' - \gamma R_1' + \rho\omega^2 P_1 = 0\,,\quad \lambda_0 P_2'' - \gamma R_2' + \rho\omega^2 P_2 = 0\,,$$
$$bR_1'' - \omega k P_2' - \omega R_2 = 0\,,\quad bR_2'' + \omega k P_1' + \omega R_1 = 0\,; \tag{8}$$

$$\mu Q_1'' + \rho\omega^2 Q_1 = 0\,,\quad \mu Q_2'' + \rho\omega^2 Q_2 = 0\,. \tag{9}$$

Here, the unknown functions P_j and R_j are combined into the one system (8) because of the connectedness of the model, and the quantity Q_j is defined by individual independent equations (9). Particular solutions of Eqs. (8) and (9) have the form

$$P_j = A_j \exp(\alpha x)\,,\quad R_j = C_j \exp(\alpha x)\,,\quad Q_j = B_j \exp(\beta x)\,,\quad j = 1, 2\,. \tag{10}$$

Substitution of (10) into (8) and (9) gives the following relations for A_j, B_j, H_j, α, and β:

$$\lambda_0\alpha^2 A_1 - \gamma\alpha H_1 + \rho\omega^2 A_1 = 0\,,\quad \lambda_0\alpha^2 A_2 - \gamma\alpha H_2 + \rho\omega^2 A_2 = 0\,,$$

$$b\alpha^2 H_1 - \omega k\alpha A_2 - \omega H_2 = 0\,, \quad b\alpha^2 H_2 + \omega k\alpha A_1 + \omega H_1 = 0\,; \tag{11}$$

$$\mu\beta^2 B_1 + \rho\omega^2 B_1 = 0\,, \quad \mu\beta^2 B_2 + \rho\omega^2 B_2 = 0\,. \tag{12}$$

From the first two equations of system (11), we obtain

$$H_1 = \frac{A_1}{\gamma}\left(\lambda_0\alpha + \frac{\rho\omega^2}{|\alpha|^2}\bar{\alpha}\right), \quad H_2 = \frac{A_2}{\gamma}\left(\lambda_0\alpha + \frac{\rho\omega^2}{|\alpha|^2}\bar{\alpha}\right), \tag{13}$$

where the overscribed bar denotes the conjunction operation. Equating the determinant of system (11) to zero, we obtain the equations

$$b\alpha^2\,(\lambda_0\alpha^2 + \rho\omega^2) = \pm\, i\omega\,(k\gamma\alpha^2 + \lambda_0\alpha^2 + \rho\omega^2)\,. \tag{14}$$

Let us introduce the dimensionless parameters

$$M_0 = \frac{k\gamma}{\lambda_0} = \frac{(3\lambda + 2\mu)^2\,\alpha_t^2 T_0}{C\rho\,(\lambda + 2\mu)}\,, \quad N_0 = \frac{b\rho\omega}{\lambda_0} \tag{15}$$

and the designations

$$\begin{gathered} A_* = \left(\frac{b\rho\omega}{\lambda_0}\right)^2 - \left(1 + \frac{k\gamma}{\lambda_0}\right)^2 = N_0^2 - (1 + M_0)^2\,, \quad B_* = 2\,\frac{b\rho\omega}{\lambda_0}\left(1 - \frac{k\gamma}{\lambda_0}\right) = 2N_0\,(1 - M_0)\,, \\ K_0 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{A_*^2 + B_*^2} + A_*}\,, \quad L_0 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{A_*^2 + B_*^2} - A_*}\,. \end{gathered} \tag{16}$$

In this case, the roots of the first equation of (14) (with the sign "+") can be represented in the form

$$\textit{if}\;\; M_0 < 1\,, \;\; \textit{then}\;\; \alpha_{1-4}^2 = \frac{\omega}{2b}\Big[i\,(1 + M_0) - N_0 \pm (K_0 + iL_0)\Big], \tag{17}$$

and the roots of the second equation of (14) (with the sign "−") can be defined as

$$\textit{if}\;\; M_0 < 1\,, \;\; \textit{then}\;\; \alpha_{5-8}^2 = \frac{\omega}{2b}\Big[-i\,(1 + M_0) - N_0 \pm (K_0 - iL_0)\Big]. \tag{18}$$

On condition that $M_0 > 1$, the roots of Eq. (14) (with the sign "+") will have the form

$$\textit{if}\;\; M_0 > 1\,, \;\; \textit{then}\;\; \alpha_{1-4}^2 = \frac{\omega}{2b}\Big[i\,(1 + M_0) - N_0 \pm (K_0 - iL_0)\Big], \tag{19}$$

and the roots of Eq. (14) (with the sign "−") will be defined as

$$\textit{if}\;\; M_0 > 1\,, \;\; \textit{then}\;\; \alpha_{5-8}^2 = \frac{\omega}{2b}\Big[-i\,(1 + M_0) - N_0 \pm (K_0 + iL_0)\Big]. \tag{20}$$

To obtain the roots α_{1-8} in explicit form, we will establish that the sign of the imaginary part of the expression for α_{1-2}^2 is larger than zero, i.e., that the following inequality is true:

$$1 + M_0 - L_0 > 0\,. \tag{21}$$

It can be written, in view of (16), as

$$2\,(1+M_0)^2 > 2L_0^2 = \sqrt{[N_0^2-(1+M_0)^2]^2+4N_0^2\,(1-M_0)^2} - N_0^2 + (1+M_0)^2\,.$$

This gives, on several simplifications, the inequality $(1+M_0)^2 > (1-M_0)^2$, which is correct because $M_0 > 0$ by definition. Inequality (21) simplifies the operation of taking the square root of a number in calculating the quantities $\alpha_1 - \alpha_8$, which allows one to obtain concrete expressions for them. For the case $M_0 < 1$, we have

$$\begin{aligned} &\alpha_1 = \alpha_{10} + i\beta_{10}\,, \quad \alpha_{10} = \sqrt{\frac{\omega}{4b}}\sqrt{\sqrt{(K_0-N_0)^2+(1+M_0+L_0)^2}+K_0-N_0}\,, \\ &\alpha_2 = -\,\alpha_1\,, \quad \beta_{10} = \sqrt{\frac{\omega}{4b}}\sqrt{\sqrt{(K_0-N_0)^2+(1+M_0+L_0)^2}-K_0-N_0}\,; \end{aligned} \tag{22}$$

$$\begin{aligned} &\alpha_3 = \alpha_{30} + i\beta_{30}\,, \quad \alpha_{30} = \sqrt{\frac{\omega}{4b}}\sqrt{\sqrt{(K_0+N_0)^2+(1+M_0-L_0)^2}-K_0+N_0}\,, \\ &\alpha_4 = -\,\alpha_3\,, \quad \beta_{30} = \sqrt{\frac{\omega}{4b}}\sqrt{\sqrt{(K_0+N_0)^2+(1+M_0-L_0)^2}+K_0+N_0}\,; \end{aligned} \tag{23}$$

$$\alpha_5 = \alpha_{10} - i\beta_{10}\,, \quad \alpha_6 = -\,\alpha_5\,, \quad \alpha_7 = \alpha_{30} - i\beta_{30}\,, \quad \alpha_8 = -\,\alpha_7\,. \tag{24}$$

In the case where $M_0 > 1$, the designations of the roots contain an asterisk at the top right:

$$\begin{aligned} &\alpha_1^* = \alpha_{10}^* + i\beta_{10}^*\,, \quad \alpha_{10}^* = \sqrt{\frac{\omega}{4b}}\sqrt{\sqrt{(K_0-N_0)^2+(1+M_0-L_0)^2}+K_0-N_0}\,, \\ &\alpha_2^* = -\,\alpha_1^*\,, \quad \beta_{10}^* = \sqrt{\frac{\omega}{4b}}\sqrt{\sqrt{(K_0-N_0)^2+(1+M_0-L_0)^2}-K_0-N_0}\,; \end{aligned} \tag{25}$$

$$\begin{aligned} &\alpha_3^* = \alpha_{30}^* + i\beta_{30}^*\,, \quad \alpha_{30}^* = \sqrt{\frac{\omega}{4b}}\sqrt{\sqrt{(K_0+N_0)^2+(1+M_0+L_0)^2}-K_0+N_0}\,, \\ &\alpha_4^* = -\,\alpha_3^*\,, \quad \beta_{30}^* = \sqrt{\frac{\omega}{4b}}\sqrt{\sqrt{(K_0+N_0)^2+(1+M_0+L_0)^2}+K_0+N_0}\,; \end{aligned} \tag{26}$$

$$\alpha_5^* = \alpha_{10}^* - i\beta_{10}^*\,, \quad \alpha_6^* = -\,\alpha_5^*\,, \quad \alpha_7^* = \alpha_{30}^* - i\beta_{30}^*\,, \quad \alpha_8^* = -\,\alpha_7^*\,. \tag{27}$$

If the dimensionless parameter $M_0 < 1$, the quantities α_{10}, α_{30}, β_{10}, and β_{30} should be determined from (22)–(24); when $M_0 > 1$, these quantities are determined from (25)–(27). By the form of the characteristic roots (22)–(27), we determine the complex-conjugate pairs that will be used in the subsequent discussion:

$$\alpha_5 = \overline{\alpha}_1\,, \quad \alpha_6 = \overline{\alpha}_2\,, \quad \alpha_7 = \overline{\alpha}_3\,, \quad \alpha_8 = \overline{\alpha}_4\,. \tag{28}$$

To obtain a general solution of system (8) in explicit form, it is necessary to determine the relations between the coefficients A_j and B_j ($j = 1, 2$) at different values of $\alpha = \alpha_k$ ($k = 1, ..., 8$). For this purpose, we will introduce the following designations:

$$\text{at} \quad \alpha = \alpha_k \Rightarrow A_j = A_j(\alpha_k), \quad H_j = H_j(\alpha_k) \quad (j = 1, 2; \quad k = 1, ..., 8), \quad A_1(\alpha_k) = D_k. \tag{29}$$

All the coefficients $A_2(\alpha_k)$ and $H_j(\alpha_k)$ are expressed in terms of the quantities D_k, which are considered as complex constants. On substitution of $\alpha = \alpha_k$ ($k = 1, ..., 8$) into (11) and (13), we determine the desired relations:

$$A_2(\alpha_{1-4}) = iA_1(\alpha_{1-4}) = iD_{1-4}, \quad A_2(\alpha_{5-8}) = -iA_1(\alpha_{5-8}) = -iD_{5-8},$$

$$H_1(\alpha_k) = \left(\lambda_0\alpha_k + \frac{\rho\omega^2}{|\alpha_k|^2}\bar{\alpha}_k\right)\frac{D_k}{\gamma} \quad (k = 1, ..., 8), \tag{30}$$

$$H_2(\alpha_{1-4}) = i\left(\frac{\lambda_0}{\gamma}\alpha_{1-4} + \frac{\rho\omega^2}{\gamma|\alpha_{1-4}|^2}\bar{\alpha}_{1-4}\right)D_{1-4}, \quad H_2(\alpha_{5-8}) = i\left(\frac{\lambda_0}{\gamma}\alpha_{5-8} + \frac{\rho\omega^2}{\gamma|\alpha_{5-8}|^2}\bar{\alpha}_{5-8}\right)D_{5-8}.$$

Now the general solution of system (8) will take the form

$$P_1(x) = \sum_{k=1}^{8} D_k \exp(\alpha_k x), \quad P_2(x) = i\sum_{k=1}^{4} D_k \exp(\alpha_k x) - i\sum_{k=1}^{4} D_{k+4} \exp(\alpha_{k+4} x), \tag{31}$$

$$R_1(x) = \sum_{k=1}^{8} D_k s_k \exp(\alpha_k x), \quad R_2(x) = i\sum_{k=1}^{4} D_k s_k \exp(\alpha_k x) - i\sum_{k=5}^{8} D_k s_k \exp(\alpha_k x), \tag{32}$$

$$s_k = \frac{\lambda_0\alpha_k}{\gamma} + \frac{\rho\omega^2\bar{\alpha}_k}{\gamma|\alpha_k|^2}.$$

To the complex-conjugate pair of characteristic roots (28) correspond the following pairs of complex-conjugate coefficients:

$$D_k = \frac{1}{2}(A_{0k} - iH_{0k}), \quad k = 1, ..., 4; \quad D_{k+4} = \bar{D}_k = \frac{1}{2}(A_{0k} + iH_{0k}). \tag{33}$$

From formulas (30)–(33) the following property follows: *the sum of the two terms in expressions (31) and (32) corresponding to the two complex-conjugate characteristic roots* α_k *and* α_{k+4} ($k = 1, ..., 4$) *is a real function*. We will demonstrate this with the example of $U_1(x)$, where the first term will involve α_1 and the second term will involve α_5:

$$D_1\exp(\alpha_1 x) + D_5\exp(\alpha_5 x) = \frac{1}{2}(A_{01} - iH_{01})(\cos\beta_{10}x + i\sin\beta_{10}x)\exp(\alpha_1 x)$$
$$+\frac{1}{2}(A_{01} + iH_{01})(\cos\beta_{10}x - i\sin\beta_{10}x)\exp(\alpha_1 x) = (A_{01}\cos\beta_{10}x + H_{01}\sin\beta_{10}x)\exp(\alpha_1 x). \tag{34}$$

For convenience, we will introduce the auxiliary constants

$$p_j = \frac{1}{\gamma}\left(\lambda_0 + \rho\omega^2/|\alpha_j|^2\right)\alpha_{j0}, \quad q_j = \frac{1}{\gamma}\left(\lambda_0 - \rho\omega^2/|\alpha_j|^2\right)\beta_{j0}, \quad j = 1, 3, \tag{35}$$

$$|\alpha_1|^2 = \frac{\omega}{2b}\sqrt{(K_0 - N_0)^2 + (1 + M_0 + L_0)^2}\,, \quad |\alpha_3|^2 = \frac{\omega}{2b}\sqrt{(K_0 + N_0)^2 + (1 + M_0 - L_0)^2}\,.$$

With the use of expression (34) and designations (35), the general solutions (31) and (32) for a plane layer are brought to the real form. If the variance x in expressions (31) and (32) is replaced by the difference $(x - h/2)$, which is more handy for the further calculations, $P_j(x)$ and $R_j(x)$ take the form

$$\begin{aligned} P_1(x) = {} & \exp \alpha_{10}\left(x - \frac{h}{2}\right)\left[A_{01}\cos\beta_{10}\left(x - \frac{h}{2}\right) + H_{01}\sin\beta_{10}\left(x - \frac{h}{2}\right)\right] \\ & + \exp\alpha_{10}\left(\frac{h}{2} - x\right)\left[A_{02}\cos\beta_{10}\left(x - \frac{h}{2}\right) - H_{02}\sin\beta_{10}\left(x - \frac{h}{2}\right)\right] \\ & + \exp\alpha_{30}\left(x - \frac{h}{2}\right)\left[A_{03}\cos\beta_{30}\left(x - \frac{h}{2}\right) + H_{03}\sin\beta_{30}\left(x - \frac{h}{2}\right)\right] \\ & + \exp\alpha_{30}\left(\frac{h}{2} - x\right)\left[A_{04}\cos\beta_{30}\left(x - \frac{h}{2}\right) - H_{04}\sin\beta_{30}\left(x - \frac{h}{2}\right)\right], \end{aligned} \tag{36}$$

$$\begin{aligned} P_2(x) = {} & \exp \alpha_{10}\left(x - \frac{h}{2}\right)\left[H_{01}\cos\beta_{10}\left(x - \frac{h}{2}\right) - A_{01}\sin\beta_{10}\left(x - \frac{h}{2}\right)\right] \\ & + \exp\alpha_{10}\left(\frac{h}{2} - x\right)\left[H_{02}\cos\beta_{10}\left(x - \frac{h}{2}\right) + A_{02}\sin\beta_{10}\left(x - \frac{h}{2}\right)\right] \\ & + \exp\alpha_{30}\left(x - \frac{h}{2}\right)\left[H_{03}\cos\beta_{30}\left(x - \frac{h}{2}\right) - A_{03}\sin\beta_{30}\left(x - \frac{h}{2}\right)\right] \\ & + \exp\alpha_{30}\left(\frac{h}{2} - x\right)\left[H_{04}\cos\beta_{30}\left(x - \frac{h}{2}\right) + A_{04}\sin\beta_{30}\left(x - \frac{h}{2}\right)\right], \end{aligned} \tag{37}$$

$$\begin{aligned} R_1(x) = {} & A_{01}\exp \alpha_{10}\left(x - \frac{h}{2}\right)\left[p_1\cos\beta_{10}\left(x - \frac{h}{2}\right) - q_1\sin\beta_{10}\left(x - \frac{h}{2}\right)\right] \\ & + H_{01}\exp\alpha_{10}\left(x - \frac{h}{2}\right)\left[p_1\sin\beta_{10}\left(x - \frac{h}{2}\right) + q_1\cos\beta_{10}\left(x - \frac{h}{2}\right)\right] \\ & - A_{02}\exp\alpha_{10}\left(\frac{h}{2} - x\right)\left[p_1\cos\beta_{10}\left(x - \frac{h}{2}\right) + q_1\sin\beta_{10}\left(x - \frac{h}{2}\right)\right] \\ & + H_{02}\exp\alpha_{10}\left(\frac{h}{2} - x\right)\left[p_1\sin\beta_{10}\left(x - \frac{h}{2}\right) - q_1\cos\beta_{10}\left(x - \frac{h}{2}\right)\right] \\ & + A_{03}\exp\alpha_{30}\left(x - \frac{h}{2}\right)\left[p_3\cos\beta_{30}\left(x - \frac{h}{2}\right) - q_3\sin\beta_{30}\left(x - \frac{h}{2}\right)\right] \\ & + H_{03}\exp\alpha_{30}\left(x - \frac{h}{2}\right)\left[p_3\sin\beta_{30}\left(x - \frac{h}{2}\right) + q_3\cos\beta_{30}\left(x - \frac{h}{2}\right)\right] \\ & - A_{04}\exp\alpha_{30}\left(\frac{h}{2} - x\right)\left[p_3\cos\beta_{30}\left(x - \frac{h}{2}\right) + q_3\sin\beta_{30}\left(x - \frac{h}{2}\right)\right] \\ & + H_{04}\exp\alpha_{30}\left(\frac{h}{2} - x\right)\left[p_3\sin\beta_{30}\left(x - \frac{h}{2}\right) - q_3\cos\beta_{30}\left(x - \frac{h}{2}\right)\right], \end{aligned} \tag{38}$$

$$R_2(x) = -A_{01} \exp \alpha_{10}\left(x-\frac{h}{2}\right)\left[p_1 \sin \beta_{10}\left(x-\frac{h}{2}\right)+q_1 \cos \beta_{10}\left(x-\frac{h}{2}\right)\right]$$

$$+H_{01} \exp \alpha_{10}\left(x-\frac{h}{2}\right)\left[p_1 \cos \beta_{10}\left(x-\frac{h}{2}\right)-q_1 \sin \beta_{10}\left(x-\frac{h}{2}\right)\right]$$

$$-A_{02} \exp \alpha_{10}\left(\frac{h}{2}-x\right)\left[p_1 \sin \beta_{10}\left(x-\frac{h}{2}\right)-q_1 \cos \beta_{10}\left(x-\frac{h}{2}\right)\right]$$

$$-H_{02} \exp \alpha_{10}\left(\frac{h}{2}-x\right)\left[p_1 \cos \beta_{10}\left(x-\frac{h}{2}\right)+q_1 \cos \beta_{10}\left(x-\frac{h}{2}\right)\right]$$

$$-A_{03} \exp \alpha_{30}\left(x-\frac{h}{2}\right)\left[p_3 \sin \beta_{30}\left(x-\frac{h}{2}\right)+q_3 \cos \beta_{30}\left(x-\frac{h}{2}\right)\right]$$

$$+H_{03} \exp \alpha_{30}\left(x-\frac{h}{2}\right)\left[p_3 \cos \beta_{30}\left(x-\frac{h}{2}\right)-q_3 \sin \beta_{30}\left(x-\frac{h}{2}\right)\right]$$

$$-A_{04} \exp \alpha_{30}\left(\frac{h}{2}-x\right)\left[p_3 \sin \beta_{30}\left(x-\frac{h}{2}\right)-q_3 \cos \beta_{30}\left(x-\frac{h}{2}\right)\right]$$

$$-H_{04} \exp \alpha_{30}\left(\frac{h}{2}-x\right)\left[p_3 \cos \beta_{30}\left(x-\frac{h}{2}\right)+q_3 \sin \beta_{30}\left(x-\frac{h}{2}\right)\right]. \tag{39}$$

The general solution of Eqs. (9) is obtained analogously:

$$Q_j(x) = B_{j1} \cos \omega \sqrt{\frac{\rho}{\mu}}\left(x-\frac{h}{2}\right)+B_{j2} \sin \omega \sqrt{\frac{\rho}{\mu}}\left(x-\frac{h}{2}\right), \quad j = 1, 2\,. \tag{40}$$

The general integrals for a thermoelastic plane layer (36)–(40) contain eight arbitrary constants A_{0j}, H_{0j} (j = 1, ..., 4) and four constants B_{j1}, B_{j2} (j = 1, 2) that are determined from the conditions set at the boundaries of the layer. The functions determined will be used for obtaining exact solutions for a thermostatic rod of triangular cross section.

The First Exact Solution. The results presented below were obtained by a special technique with the use of ξ variables [6] in the following way.

Let $\mathbf{r}_0$ and $\mathbf{r}$ be the radii-vectors of any pole and an arbitrary point inside the cross section of a rod Ω, $\mathbf{r}_k$ (k = 1, 2, 3) be the radii-vectors of the vertices of the right triangle Ω of height h, and the auxiliary variables ξ and ξ_k be determined by the formulas

$$\xi = (\mathbf{r}-\mathbf{r}_0)\,\mathbf{n}\,, \quad \xi_k = (\mathbf{r}-\mathbf{r}_k)\,\mathbf{n}_k\,, \quad k = 1, 2, 3\,, \tag{41}$$

where $\mathbf{n}$ is a unit vector, $\mathbf{n}_j$ are internal unit normals to the sides of the triangle Ω, the vertices and sides of which are numbered in a counterclockwise direction. With such variables ξ_k, the equations for the sides of the triangle will have the form $\xi_1 = 0$, $\xi_2 = 0$, and $\xi_3 = 0$. For the points $(x, y) \in \Omega$, the strict inequalities $\xi_1 > 0$, $\xi_2 > 0$, and $\xi_3 > 0$ are true. The variables ξ and ξ_k and the normals $\mathbf{n}_k$ in the plane (x, y) possess the following properties that will be used in the subsequent discussion:

$$\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3 = 0\,, \quad \mathbf{n}_1\mathbf{n}_2 = \mathbf{n}_1\mathbf{n}_3 = \mathbf{n}_2\mathbf{n}_3 = -1/2\,,$$
$$\mathbf{n}_1\times\mathbf{n}_2\,\big|_z = \mathbf{n}_2\times\mathbf{n}_3\,\big|_z = \mathbf{n}_3\times\mathbf{n}_1\,\big|_z = \sqrt{3}/2\,, \quad \xi_1+\xi_2+\xi_3 = h\,; \tag{42}$$

$$\textit{if} \quad F = F(\xi) \in C^2(\Omega)\,, \quad \textit{then} \quad F_x = F'(\xi)\,n_x\,, \quad F_y = F'(\xi)\,n_y\,,$$

$$F_{xx} = F''(\xi)\, n_x^2\,, \quad F_{xy} = F''(\xi)\, n_x n_y\,, \quad F_{yy} = F''(\xi)\, n_y^2\,, \quad \Delta F = F''(\xi)\,. \tag{43}$$

Using $P_j(x)$, $Q_j(x)$, and $R_j(x)$, obtained from (36)–(40), where x should be replaced by ξ, we will obtain the following partial solution of system (6) [7]:

$$U_j(x, y) = P_j(\xi)\, n_x - Q_j(\xi)\, n_y\,, \quad V_j(x, y) = P_j(\xi)\, n_y + Q_j(\xi)\, n_x\,, \quad T_j(x, y) = R_j(\xi)\,, \quad j = 1, 2\,. \tag{44}$$

The functions U_j, V_j, and T_j differ radically from each other in structure. This is explained by the fact that (U_j, V_j) is a vector function and T_j is a scalar function, and the change from x to the variable ξ is equivalent to a rotation of the coordinate system. In this case, the vector functions are rearranged by the vector-algebra laws and the scalar functions remain unchanged; therefore, the functions (U_j, V_j) involve the projections of the normal vector n_x and n_y, accounting for the above-indicated rotation, and these projections are not involved in T_j in such a form. In the subsequent discussion, the following two properties will be used.

Property 1. *If the functions $P_j(x)$, $Q_j(x)$, and $R_j(x)$ used in expressions (44) are solutions of systems (8) and (9), i.e., have the form of (36)–(40), U_j, V_j, and T_j determined from (44) satisfy all the differential equations of (6).*

Property 2. *The functions $Q_j(\xi)$ in (44), representing partial solutions of Eqs. (9), can be selected independently of the partial solutions $P_j(\xi)$ and $R_j(\xi)$.*

To prove these properties, we substitute U_j, V_j, *and* T_j determined from (44) into the first and third equations of (6); analogous actions are performed for the second and fourth equations. Using the partial derivatives determined from (43), we obtain

$$\begin{gathered}\lambda_0 (P_j'' n_x^3 - Q_j'' n_x^2 n_y) + (\lambda + \mu)(P_j'' n_x n_y^2 + Q_j'' n_x^2 n_y) \\ + \mu (P_j'' n_x n_y^2 - Q_j'' n_y^3) - \rho\omega^2 (P_j n_x - Q_j n_y) = 0\,, \quad j = 1, 2\,; \\ bR_1'' - k\omega (P_2' n_x^2 - Q_2' n_x n_y) - k\omega (P_2' n_y^2 + Q_2' n_x n_y) - \omega R_2 = 0\,.\end{gathered} \tag{45}$$

The vector (n_x, n_y) is a unit vector; therefore, the last equation of (45), on simplification, becomes identical to the third equation of (8). In the first equation of (45), all the terms ahead of P_j and Q_j are regrouped in the following way:

$$P_j'' n_x (\lambda_0 n_x^2 + (\lambda + \mu)\, n_y^2 + \mu n_y^2) + \rho\omega^2 n_x P_j - Q_j'' n_y (\lambda_0 n_x^2 - (\lambda + \mu)\, n_x^2 + \mu n_y^2) - \rho\omega^2 n_y Q_1 = 0\,. \tag{46}$$

The coefficients of P_j'' and Q_j'' are rearranged by the formulas

$$(\lambda_0 n_x^2 + (\lambda + \mu)\, n_y^2 + \mu n_y^2) = \lambda_0 n_x^2 + \lambda_0 n_y^2 = \lambda_0\,, \quad (\lambda_0 n_x^2 - (\lambda + \mu)\, n_x^2 + \mu n_y^2) = \mu n_x^2 + \mu n_y^2 = \mu\,, \tag{47}$$

in view of which Eq. (46) can be written as

$$n_x (\lambda_0 P_j'' - \gamma R_j' + \rho\omega^2 P_j) - n_y (\mu Q_j'' + \rho\omega^2 Q_j) = 0\,. \tag{48}$$

Since P_j, Q_j, and R_j satisfy Eqs. (8) and (9) in their structure, the parenthetical expressions (48) are equal to zero. It follows herefrom that the above-indicated properties are proved. If the variable ξ on the right side of expressions (44) is replaced by any variable ξ_k determined from (41), the newly obtained U_j, V_j, *and* T_j will also satisfy system (6).

For the sake of convenience of representation of the exact solution, we introduce the functions

$$\begin{gathered}P_j^{(a)}(\xi_1) = P_j(\xi_1) - P_j(h - \xi_1)\,, \quad R_j^{(s)}(\xi_1) = R_j(\xi_1) + R_j(h - \xi_1)\,, \quad j = 1, 2\,; \\ Q_j^{(s)}(\xi_1) = B_{j1} \cos \omega \sqrt{\frac{\rho}{\mu}} \left(\xi_1 - \frac{h}{2}\right), \quad Q_j^{(a)}(\xi_1) = B_{j2} \sin \omega \sqrt{\frac{\rho}{\mu}} \left(\xi_1 - \frac{h}{2}\right).\end{gathered} \tag{49}$$

The functions $P_j^{(s)}(\xi_1)$ and $R_j^{(a)}(\xi_1)$ are introduced in a similar way. If $P_j(\xi_1)$ and $R_j(\xi_1)$ involve eight constants, $P_j^{(s)}(\xi_1)$ and $R_j^{(a)}(\xi_1)$ involve only four constants, which will be denoted as E_1–E_4; in this case,

$$E_1 = 2\,(A_{01} - A_{02})\,, \quad E_2 = 2\,(H_{01} - H_{02})\,, \quad E_3 = 2\,(A_{03} - A_{04})\,, \quad E_4 = 2\,(H_{03} - H_{04})\,. \tag{50}$$

The functions $P_j^{(a)}(\xi_1)$ and $R_j^{(s)}(\xi_1)$ have the form

$$\begin{aligned} P_1^{(a)}(\xi) = {} & E_1 \cos \beta_{10}\left(\xi - \frac{h}{2}\right)\sinh \alpha_{10}\left(\xi - \frac{h}{2}\right) + E_2 \sin \beta_{10}\left(\xi - \frac{h}{2}\right)\cosh \alpha_{10}\left(\xi - \frac{h}{2}\right) \\ & + E_3 \cos \beta_{30}\left(\xi - \frac{h}{2}\right)\sinh \alpha_{30}\left(\xi - \frac{h}{2}\right) + E_4 \sin \beta_{30}\left(\xi - \frac{h}{2}\right)\cosh \alpha_{30}\left(\xi - \frac{h}{2}\right), \end{aligned} \tag{51}$$

$$\begin{aligned} P_2^{(a)}(\xi) = {} & E_2 \cos \beta_{10}\left(\xi - \frac{h}{2}\right)\sinh \alpha_{10}\left(\xi - \frac{h}{2}\right) - E_1 \sin \beta_{10}\left(\xi - \frac{h}{2}\right)\cosh \alpha_{10}\left(\xi - \frac{h}{2}\right) \\ & + E_4 \cos \beta_{30}\left(\xi - \frac{h}{2}\right)\sinh \alpha_{30}\left(\xi - \frac{h}{2}\right) - E_3 \sin \beta_{30}\left(\xi - \frac{h}{2}\right)\cosh \alpha_{30}\left(\xi - \frac{h}{2}\right), \end{aligned} \tag{52}$$

$$\begin{aligned} R_1^{(s)}(\xi) = {} & (E_1 p_1 + E_2 q_1) \cos \beta_{10}\left(\xi - \frac{h}{2}\right)\cosh \alpha_{10}\left(\xi - \frac{h}{2}\right) + (E_2 p_1 - E_1 q_1) \sin \beta_{10}\left(\xi - \frac{h}{2}\right)\sinh \alpha_{10}\left(\xi - \frac{h}{2}\right) \\ & + (E_3 p_3 + E_4 q_3) \cos \beta_{30}\left(\xi - \frac{h}{2}\right)\cosh \alpha_{30}\left(\xi - \frac{h}{2}\right) + (E_4 p_3 - E_3 q_3) \sin \beta_{30}\left(\xi - \frac{h}{2}\right)\sinh \alpha_{30}\left(\xi - \frac{h}{2}\right), \end{aligned} \tag{53}$$

$$\begin{aligned} R_2^{(s)}(\xi) = {} & (E_2 p_1 - E_1 q_1) \cos \beta_{10}\left(\xi - \frac{h}{2}\right)\cosh \alpha_{10}\left(\xi - \frac{h}{2}\right) - (E_1 p_1 + E_2 q_1) \sin \beta_{10}\left(\xi - \frac{h}{2}\right)\cosh \alpha_{10}\left(\xi - \frac{h}{2}\right) \\ & + (E_4 p_3 - E_3 q_3) \cos \beta_{30}\left(\xi - \frac{h}{2}\right)\cosh \alpha_{30}\left(\xi - \frac{h}{2}\right) - (E_3 p_3 + E_4 q_3) \sin \beta_{30}\left(\xi - \frac{h}{2}\right)\sinh \alpha_{30}\left(\xi - \frac{h}{2}\right). \end{aligned} \tag{54}$$

The index (s) or (a) at the top right of the quantities means that the function is symmetric or antisymmetric relative to the point $\xi = h/2$; therefore, the following equalities are fulfilled for them:

$$\begin{aligned} & P_j^{(a)}(\xi_1) = -P_j^{(a)}(h - \xi_1)\,, \quad R_j^{(s)}(\xi_1) = R_j^{(s)}(h - \xi_1)\,, \\ & P_j^{(a)\prime}(\xi_1) = P_j^{(a)\prime}(h - \xi_1)\,, \quad R_j^{(s)\prime}(\xi_1) = -R_j^{(s)\prime}(h - \xi_1)\,. \end{aligned} \tag{55}$$

The solution of problem (6) with boundary conditions (3) represents the sums

$$\begin{aligned} U_j(x, y) &= P_j^{(a)}(\xi_1)\, n_{1x} + P_j^{(a)}(\xi_2)\, n_{2x} + P_j^{(a)}(\xi_3)\, n_{3x} - Q_j^{(s)}(\xi_1)\, n_{1y} - Q_j^{(s)}(\xi_2)\, n_{2y} - Q_j^{(s)}(\xi_3)\, n_{3y}\,, \\ V_j(x, y) &= P_j^{(a)}(\xi_1)\, n_{1y} + P_j^{(a)}(\xi_2)\, n_{2y} + P_j^{(a)}(\xi_3)\, n_{3y} + Q_j^{(s)}(\xi_1)\, n_{1x} + Q_j^{(s)}(\xi_2)\, n_{2x} + Q_j^{(s)}(\xi_3)\, n_{3x}\,, \\ T_j(x, y) &= R_j^{(s)}(\xi_1) + R_j^{(s)}(\xi_2) + R_j^{(s)}(\xi_3)\,, \quad j = 1, 2\,. \end{aligned} \tag{56}$$

Because of properties 1 and 2, the values of U_j, V_j, *and* T_j determined from (56) satisfy Eqs. (6). It remains to fulfill the boundary conditions (3), which will be preliminarily rearranged. The normal component of the displacements in Γ is defined as

$$u_{\mathrm{n}}\big|_\Gamma = (u n_x + v n_y)\big|_\Gamma \quad \text{or} \quad (U_j n_x + V_j n_y)\big|_\Gamma = u_{j0}\,, \quad j = 1, 2\,. \tag{57}$$

In the problems being considered, it is assumed that all the analytical dependences are equivalent in relation to the sides of the right triangle; therefore, it will be sufficient to fulfill the boundary conditions at any one of its sides, e.g., at the side $\xi_3 = 0$. In this case, at the two sides of the triangle, the boundary conditions will be fulfilled automatically at $\xi_1 = 0$ and $\xi_2 = 0$. For the points (x, y) at the triangle side $\xi_3 = 0$, the variables ξ_1 and ξ_2 are related, at $\xi_3 = 0$, by the relation

$$\xi_1 + \xi_2 = h \, . \tag{58}$$

Substituting U_j and V_j, determined from (56), into (57) at $\xi_3 = 0$ and using (58), we obtain

$$\begin{gathered}\left[P_j^{(a)}(\xi_1)\,(\mathbf{n}_1\mathbf{n}_3) + P_j^{(a)}(h-\xi_1)\,(\mathbf{n}_2\mathbf{n}_3)\right] + P_j^{(a)}(0) + \\ + \left[Q_j^{(s)}(\xi_1)\,(\mathbf{n}_1\times\mathbf{n}_3)\,|_z + Q_j^{(s)}(h-\xi_1)\,(\mathbf{n}_2\times\mathbf{n}_3)\,|_z\right] = u_{j0}\,, \quad j = 1, 2\,.\end{gathered} \tag{59}$$

Using properties (42) and (55), it can be shown that all the terms with variables ξ_1 in both square brackets cancel one another and, therefore,

$$P_j^{(a)}(0) = u_{j0}\,, \quad j = 1, 2\,. \tag{60}$$

The boundary condition for the shear stress, determined from (13), can be defined, in accordance with (1), as

$$\tau_n\,|_\Gamma = 2\mu\gamma_n\,|_\Gamma = \left(\frac{\partial u_\tau}{\partial n} + \frac{\partial u_n}{\partial \tau}\right)\Bigg|_\Gamma \,. \tag{61}$$

If the normal stress in Γ is determined by the unit vector $\mathbf{n} = (n_x, n_y)$, the shear stress in Γ in the plane problem will be determined by the unit vector $\tau = (-n_y, n_x)$. Therefore, the shear component of the displacement vector in Γ is determined from the equality

$$u_\tau\,|_\Gamma = (-\,u n_y + v n_x)\,|_\Gamma \,. \tag{62}$$

The normal component u_n in Γ, determined from (3), is assumed to be constant at different points of the boundary; therefore, the expressions for the shear γ_n can be simplified:

$$2\gamma_n\,|_\Gamma = \frac{\partial u_\tau}{\partial n}\Bigg|_\Gamma = \left(-\frac{\partial u}{\partial n}\,n_y + \frac{\partial v}{\partial n}\,n_x\right)\Bigg|_\Gamma \,. \tag{63}$$

Now the boundary conditions for τ_n, determined from (3), take the form

$$\mu\left(-\frac{\partial U_j}{\partial n}\,n_y + \frac{\partial V_j}{\partial n}\,n_x\right)\Bigg|_\Gamma = \tau_{j0}\,, \quad j = 1, 2\,. \tag{64}$$

Substitution of U_j and V_j determined from (56) into (64) at $\xi_3 = 0$ gives

$$\begin{gathered}-\left[P_j^{(a)\prime}(\xi_1)\,(\mathbf{n}_1\times\mathbf{n}_3)\,|_z\,(\mathbf{n}_1\mathbf{n}_3) + P_j^{(a)\prime}(h-\xi_1)\,(\mathbf{n}_2\times\mathbf{n}_3)\,|_z\,(\mathbf{n}_2\mathbf{n}_3)\right] \\ + \left[Q_j^{(s)\prime}(\xi_1)\,(\mathbf{n}_1\mathbf{n}_3)^2 + Q_j^{(s)\prime}(h-\xi_1)\,(\mathbf{n}_2\mathbf{n}_3)^2\right] + Q_j^{(s)\prime}(0) = \frac{\tau_{j0}}{\mu}\,, \quad j = 1, 2\,.\end{gathered} \tag{65}$$

In accordance with (42) and (55), the expressions in both square brackets are equal to zero and, therefore, from (65) we obtain

$$Q_j^{(s)\prime}(0) = \frac{\tau_{j0}}{\mu}, \quad j = 1, 2 . \tag{66}$$

From (66) it follows that

$$B_{j1} = \frac{\tau_{j0}}{\omega\sqrt{\mu\rho}} \Big/ \sin\left(\frac{1}{2}\omega h\sqrt{\frac{\rho}{\mu}}\right), \quad j = 1, 2 , \quad \omega \neq \omega^* = 2\frac{\pi n}{h}\sqrt{\frac{\mu}{\rho}} . \tag{67}$$

It is seen from this expression that a resonance between the shear deformations can arise in a thermoelastic material at $\omega = \omega^*$. It remains to fulfill the last boundary condition determined from (3) for the heat flow at the side $\xi_3 = 0$. This condition, on substitution of $T_j(x, y)$ from (56), takes the form

$$\left[R_j^{(s)\prime}(\xi_1)(\mathbf{n}_1\mathbf{n}_3) + R_j^{(s)\prime}(h - \xi_1)(\mathbf{n}_2\mathbf{n}_3)\right] + R_j^{(s)\prime}(0) = q_{j0} , \quad j = 1, 2 . \tag{68}$$

Using (42) and (56), we prove that the bracketed expression is equal to zero and, from (68), obtain the equation

$$R_j^{(s)\prime}(0) = q_{j0} , \quad j = 1, 2 . \tag{69}$$

From the closed system of equations (60) and (69), we determine the constants E_1–E_4.

The Second Exact Solution. Let the boundary conditions be defined by (4); then the second condition, determined from (4), can be rearranged to the more suitable form

$$u_\tau\big|_\Gamma = (-un_y + vn_x)\big|_\Gamma \quad \text{or} \quad (-U_jn_y + V_jn_x)\big|_\Gamma = v_{j0} , \quad j = 1, 2 . \tag{70}$$

The solution of problem (6) with boundary conditions (4) will have the form

$$\begin{gathered} U_j(x, y) = P_j^{(s)}(\xi_1)\, n_{1x} + P_j^{(s)}(\xi_2)\, n_{2x} + P_j^{(s)}(\xi_3)\, n_{3x} - Q_j^{(a)}(\xi_1)\, n_{1y} - Q_j^{(a)}(\xi_2)\, n_{2y} - Q_j^{(a)}(\xi_3)\, n_{3y} , \\ V_j(x, y) = P_j^{(s)}(\xi_1)\, n_{1y} + P_j^{(s)}(\xi_2)\, n_{2y} + P_j^{(s)}(\xi_3)\, n_{3y} + Q_j^{(a)}(\xi_1)\, n_{1x} + Q_j^{(a)}(\xi_2)\, n_{2x} + Q_j^{(a)}(\xi_3)\, n_{3x} , \\ T_j(x, y) = R_j^{(a)}(\xi_1) + R_j^{(a)}(\xi_2) + R_j^{(a)}(\xi_3) , \quad j = 1, 2 . \end{gathered} \tag{71}$$

Substitution of (71) into (70) at $\xi_3 = 0$ gives

$$\begin{gathered} -\left[P_j^{(s)}(\xi_1)(\mathbf{n}_1 \times \mathbf{n}_3)\big|_z + P_j^{(s)}(h - \xi_1)(\mathbf{n}_2 \times \mathbf{n}_3)\big|_z\right] \\ + \left[Q_j^{(a)}(\xi_1)(\mathbf{n}_1\mathbf{n}_3) + Q_j^{(a)}(h - \xi_1)(\mathbf{n}_2\mathbf{n}_3)\right] + Q_j^{(a)}(0) = v_{j0} , \quad j = 1, 2 . \end{gathered} \tag{72}$$

It can be shown, using (42) and (55), that both bracketed expressions containing ξ_1 are equal to zero; therefore, from (72) it follows that

$$Q_j^{(a)}(0) = v_{j0} , \quad j = 1, 2 . \tag{73}$$

Hence, using the expression for $Q_j^{(a)}(\xi_1)$, from (49) we obtain

$$B_{j2} = -v_{j0}/\sin\left(\frac{1}{2}\omega h\sqrt{\frac{\rho}{\mu}}\right), \quad j = 1, 2 , \quad \omega \neq \omega^* = 2\frac{\pi}{h}\sqrt{\frac{\mu}{\rho}} . \tag{74}$$

In order that the first boundary condition determined from (4) can be used, it should be rearranged:

$$\sigma_{\mathrm{n}}\,|_{\Gamma}=\lambda_0\left.\frac{\partial u_{\mathrm{n}}}{\partial n}\right|_{\Gamma}=\lambda_0\frac{\partial}{\partial n}(un_x+vn_y)\,|_{\Gamma}\quad\text{or}\quad\frac{\partial}{\partial n}(U_jn_x+V_jn_y)\,|_{\Gamma}=\frac{\sigma_{j0}}{\lambda_0}\,,\quad j=1,2\,. \tag{75}$$

Substitution of (71) into (75) and then into the third condition determined from (4) gives

$$\left[P_j^{(\mathrm{s})\prime}(\xi_1)\,(\mathbf{n}_1\mathbf{n}_3)^2+P_j^{(\mathrm{s})\prime}(h-\xi_1)\,(\mathbf{n}_2\mathbf{n}_3)^2\right]+P_j^{(\mathrm{s})\prime}(0)$$

$$+\left[Q_j^{(\mathrm{a})\prime}(\xi_1)\,(\mathbf{n}_1\times\mathbf{n}_3)\,|_z\,(\mathbf{n}_1\mathbf{n}_3)+Q_j^{(\mathrm{a})\prime}(h-\xi_1)\,(\mathbf{n}_2\times\mathbf{n}_3)\,|_z\,(\mathbf{n}_2\mathbf{n}_3)\right]=v_{j0}\,,\quad j=1,2\,; \tag{76}$$

$$\left[R_j^{(\mathrm{a})}(\xi_1)+R_j^{(\mathrm{a})}(h-\xi_1)\right]+R_j^{(\mathrm{a})}(0)=T_{j0}\,,\quad j=1,2\,. \tag{77}$$

Since the functions $P_j^{(\mathrm{s})}(\xi_1)$, $Q_j^{(\mathrm{a})}(\xi_1)$, and $R_j^{(\mathrm{a})}(\xi_1)$ possess properties (55), the bracketed expressions (76) and (77) are equal to zero; therefore,

$$P_j^{(\mathrm{s})\prime}(0)=v_{j0}\,,\quad R_j^{(\mathrm{a})}(0)=T_{j0}\,,\quad j=1,2\,. \tag{78}$$

The functions $P_j^{(\mathrm{s})}(\xi_1)$ and $R_j^{(\mathrm{a})}(\xi_1)$ contain four constants and have the following form:

$$E_5=2\,(A_{01}+A_{02})\,,\quad E_6=2\,(H_{01}+H_{02})\,,\quad E_7=2\,(A_{03}+A_{04})\,,\quad E_8=2\,(H_{03}+H_{04})\,, \tag{79}$$

$$P_1^{(\mathrm{s})}(\xi)=E_5\cos\beta_{10}\left(\xi-\frac{h}{2}\right)\cosh\alpha_{10}\left(\xi-\frac{h}{2}\right)+E_6\sin\beta_{10}\left(\xi-\frac{h}{2}\right)\sinh\alpha_{10}\left(\xi-\frac{h}{2}\right)$$

$$+E_7\cos\beta_{30}\left(\xi-\frac{h}{2}\right)\cosh\alpha_{30}\left(\xi-\frac{h}{2}\right)+E_8\sin\beta_{30}\left(\xi-\frac{h}{2}\right)\sinh\alpha_{30}\left(\xi-\frac{h}{2}\right), \tag{80}$$

$$P_2^{(\mathrm{s})}(\xi)=E_6\cos\beta_{10}\left(\xi-\frac{h}{2}\right)\cosh\alpha_{10}\left(\xi-\frac{h}{2}\right)-E_5\sin\beta_{10}\left(\xi-\frac{h}{2}\right)\sinh\alpha_{10}\left(\xi-\frac{h}{2}\right)$$

$$+E_8\cos\beta_{30}\left(\xi-\frac{h}{2}\right)\cosh\alpha_{30}\left(\xi-\frac{h}{2}\right)-E_7\sin\beta_{30}\left(\xi-\frac{h}{2}\right)\sinh\alpha_{30}\left(\xi-\frac{h}{2}\right), \tag{81}$$

$$R_1^{(\mathrm{a})}(\xi)=(E_5p_1+E_6q_1)\cos\beta_{10}\left(\xi-\frac{h}{2}\right)\sinh\alpha_{10}\left(\xi-\frac{h}{2}\right)+(E_6p_1-E_5q_1)\sin\beta_{10}\left(\xi-\frac{h}{2}\right)\cosh\alpha_{10}\left(\xi-\frac{h}{2}\right)$$

$$+(E_7p_3+E_8q_3)\cos\beta_{30}\left(\xi-\frac{h}{2}\right)\sinh\alpha_{30}\left(\xi-\frac{h}{2}\right)+(E_8p_3-E_7q_3)\sin\beta_{30}\left(\xi-\frac{h}{2}\right)\cosh\alpha_{30}\left(\xi-\frac{h}{2}\right), \tag{82}$$

$$R_2^{(\mathrm{a})}(\xi)=(E_6p_1-E_5q_1)\cos\beta_{10}\left(\xi-\frac{h}{2}\right)\sinh\alpha_{10}\left(\xi-\frac{h}{2}\right)-(E_5p_1+E_6q_1)\sin\beta_{10}\left(\xi-\frac{h}{2}\right)\cosh\alpha_{10}\left(\xi-\frac{h}{2}\right)$$

$$+(E_8p_3-E_7q_3)\cos\beta_{30}\left(\xi-\frac{h}{2}\right)\sinh\alpha_{30}\left(\xi-\frac{h}{2}\right)-(E_7p_3+E_8q_3)\sin\beta_{30}\left(\xi-\frac{h}{2}\right)\cosh\alpha_{30}\left(\xi-\frac{h}{2}\right). \tag{83}$$

From the closed linear system (78), we determine the four constants E_5–E_8 that complete the construction of the second exact solution.

We will not present both exact solutions in explicit form because they are cumbersome. For the same reason, we failed to strictly prove that the determinants of system (60), (69), and system (78) are not equal to zero. However, this can be done in a particular case for a nonconnected model. Moreover, numerical experiments have shown that the indicated determinants are not equal to zero in the general case, too.

The two exact solutions obtained should be used in the following way. From (5), the analytical expressions for the displacements u and v and the temperature T, which allow for the differentiation over all the variables for determining the velocities, deformations, and stresses, are determined. The amplitudes U_j, V_j, and T_j are determined from (56) for the first exact solution with boundary conditions (3) or from (71) for the second exact solution with boundary conditions (4). The expressions for $P_j^{(s)}$, $P_j^{(a)}$, $R_j^{(s)}$, and $R_j^{(a)}$ are determined from (51)–(54), the constants E_1–E_4 are determined by solving the linear system (60), (69), and the constants E_5–E_8 are determined from system (78). In the process of numerical realization of the solutions, all the operations should be carried out in the reverse order: first, the constants E_1–E_4 are determined from the solution of the linear system (60), (69) and the constants E_5–E_8 are determined from system (78), and then the functions $P_j^{(s)}$, $P_j^{(a)}$, $R_j^{(s)}$, and $R_j^{(a)}$ are determined from (51)–(54), and so on. At $M_0 > 1$, in both exact solutions (56) and (71), α_k^* determined from (25)–(27) should be used instead of α_k.

Conclusions. The exact solutions obtained by us show that harmonic vibrations of three types can arise in a thermoelastic rod: a shear wave propagating with a velocity $v_\mu = \sqrt{\mu/\rho}$ and two longitudinal waves propagating with velocities $v_T = \omega/\beta_{10}$ and $v_e = \omega/\beta_{30}$. Under the actions along the normal to the boundaries of the rod, there arise two longitudinal waves. The first of them arising at $M_0 \to 0$ (when the connectedness of the problem disappears) is a temperature wave, and the second wave is a longitudinal mechanical wave. At $M_0 > 0$ (in the connected problem), the temperature wave v_T, as a secondary effect, causes longitudinal deformations and the longitudinal wave v_e influences the temperature field. The temperature properties of the material, in accordance with the formulas for β_{10} and β_{30}, influence the velocities of the longitudinal waves v_T and v_e. The shear actions give rise to shear waves; in this case, a resonance is possible at a frequency $\omega = 2\pi n\sqrt{\mu/\rho}/h$. The temperature field and the shear waves do not influence each other. The connectedness of the thermoelastic problem in the two given exact solutions is determined by the dimensionless parameter M_0. The analytical solutions of both problems are substantially dependent on M_0, the threshold value of which is $M_0 = 1$. By the values of the thermophysical quantities taken from the reference work [8], one can calculate M_0, e.g., $M_0 = 0.02$ for aluminum and $M_0 = 0.17$ for bronze. If the formulation of the problem requires that the calculation error be smaller than the parameter M_0 or comparable with it, the connectedness should be taken into account.

NOTATION

A_*, B_*, M_0, N_0, K_0, L_0, dimensionless parameters; A_{0k}, B_{jk}, H_{0k}, D_k, E_k, constant coefficients; b, thermal diffusivity; C, specific heat capacity; $C^{(2)}$, space of twice differentiable functions; e_{ij}, σ_{ij}, deformation and stress tensors; $F(\xi)$, auxiliary differentiable function; h, height of a triangle; i, imaginary unit; k, connectivity coefficient; $\mathbf{n}_k$, internal unit normals to the sides of the triangle; (n_x, n_y) and (n_{kx}, n_{ky}), Cartesian projections of the unit vectors; q_{j0}, T_{j0}, u_j, v_{j0}, σ_{j0}, τ_{j0} ($j = 1, 2$), definite constants; p_k, q_k, auxiliary constants; $\mathbf{r}_0$, $\mathbf{r}$, $\mathbf{r}_k$, radii-vectors of any pole, an auxiliary point in Ω, and the vertices of the triangle at the cross section of a rod; P_j, Q_j, R_j, one-dimensional solution; U_j, V_j, T_j, conditional amplitudes of the harmonic oscillations of the displacement vectors u and v and the temperature T; t_0, conditional time of the whole process; u_n, u_τ, normal and tangential components of the displacement vector in Γ; v_μ, velocity of a shear elastic wave; v_e, velocity of a longitudinal elastic wave; v_T, velocity of a temperature wave; x, y, and t, Cartesian coordinates and time; α, β, characteristic roots; α_{j0}, β_{i0}, real and imaginary parts of the characteristic roots; Γ, boundary of the right triangle; δ_{ij}, unit tensor; Δ, Laplace operator; λ, μ, Lamé constants; ξ, ξ_k, auxiliary variables; ρ, density; δ_n and τ_n, normal and tangential components of the stress tensor in Γ; ω, frequency of harmonic vibrations; Ω, region of the right triangle — cross section of the rod. Subscripts: e, elasticity; n, normal.

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